Operator Matrix Representations of Inner and Outer Inverses

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Outline of the talk

- With a motivation, we first discuss inner and outer inverses.
- We next see matrix representations of inner and outer inverses.

Notation

Let X and Y be (complex) Banach spaces and let $\mathcal{B}(X, Y)$ be the set of bounded linear operators from X to Y. If $X = Y$, then we just write $\mathcal{B}(X)$.

We will write $I_X \in \mathcal{B}(X)$ for the identity operator $I_X x = x$, dropping the subscript when the context is clear, and $O \in \mathcal{B}(X, Y)$ for the null operator $Ox = 0$.

Let $A \in \mathcal{B}(X, Y)$, if there is an operator $B \in \mathcal{B}(Y, X)$ such that $AB = I_Y$ and $BA = I_X$, then we say that A is invertible with inverse $A^{-1} := B$.

Let us denote $\mathcal{N}(A) := \{x : Ax = O\}$ the null space of A and $\mathcal{R}(A) := \{Ax : x \in X\}$ the range of A.

Solution of an Operator Equation

We are interested in the following problem: given $A \in \mathcal{B}(X)$ and $y \in X$, find $x \in X$ such that

$$Ax = y. \tag{1}$$

Of course, if A is invertible, we have $x = A^{-1}y$. Thus, we are interested in solving equation (1) for the case where A is not invertible.

Throughtout we will suppose $A \in \mathcal{B}(X)$ is not invertible.

We say A is 1-1 if $\mathcal{N}(A) = \{0\}$ and A is onto if $\mathcal{R}(A) = X$. It is a consequence of the closed graph theorem that an operator is invertible if and only if it is 1-1 and onto.

In order to give a condition for being able to find a solution to (1), we introduce complemented subspaces. Let M be a closed subspace of X. If there exists a closed subspace N such that $X = M \oplus N$, then we say that M is complemented with complement N. Here, $X = M \oplus N$, then we say that M is complemented with complement N. Here, $X = M \oplus N$ means that $M \cap N = \{0\}$ and for every $x \in X$, there exists (unique) $u \in M$ and $v \in N$ such that $x = u + v$.

It is clear that an invertible operator has a closed range and complemented "range and null spaces". We are working with a non-invertible operator A, and in a sort of "generalization", we will require $\mathcal{R}(A)$ to be closed and complemented ; and $\mathcal{N}(A)$ to be complemented. Thus, suppose $\mathcal{R}(A)$ and $\mathcal{N}(A)$ are closed and complemented with complements M and N respectively.

Operator Matrix Representation

We can represent A in the following form:

$$A : \begin{bmatrix} N \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ M \end{bmatrix}. \tag{2}$$

Notice that for the reduction

$$A_1 := A|_N : N \to \mathcal{R}(A)$$

(defined by $A_1 x = Ax$ for every $x \in N$) we have $A_1 \in \mathcal{B}(N, \mathcal{R}(A))$

$$\begin{aligned} \mathcal{N}(A_1) &= \mathcal{N}(A) \cap N = \{0\} \\ \mathcal{R}(A_1) &= \mathcal{R}(A) \end{aligned}$$

and thus, A_1 is invertible.

Projection

Recall P is a projection if $P = P^2$, and in this case we have $Px = x$ for every $x \in \mathcal{R}(P)$.

Let P be a projection onto $\mathcal{R}(A)$, and let $B := A_1^{-1}P \in \mathcal{B}(X)$. Then,

$$ABA = AA^{-1}PA = A. \tag{3}$$

It follows that AB is a projection onto $\mathcal{R}(A)$:

$$(AB)^2 = ABAB = AB,$$
$$\mathcal{R}(A) = \mathcal{R}(ABA) \subseteq \mathcal{R}(AB) \subseteq \mathcal{R}(A).$$

Thus, if $y \in \mathcal{R}(A)$, then $ABy = y$. Hence, taking $x = By$ we have

$$Ax = ABy = y,$$

that is, $x = By$ is a solution for equation (1).

Using (3) it is also easily verified that, for $z \in X$ arbitrary,

$$By + (I - BA)z$$

is also a solution for equation (1).

The operator B constructed above satisfies $A = ABA$. This was one of the keys for finding a solution to (1), and it deserves a name:

Definition : Let $A \in \mathcal{B}(X)$, if there exists some $B \in \mathcal{B}(X)$ such that $A = ABA$ holds, then B is called an inner inverse for A, and we say that A is inner invertible.

We have shown that if $A \in \mathcal{B}(X)$ has closed range ; and "complemented range and null spaces", then there exists an inner inverse $B \in \mathcal{B}(X)$ for A.

Inner Inverse for A

Now we are interested in matrix forms for A and B. Recalling representation (2), we write the following matrix form:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} : \begin{bmatrix} N \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ M \end{bmatrix}.$$

We have shown above that $A_{11} : N \to \mathcal{R}(A)$ is invertible. Now, since $Ax = 0$ for every $x \in \mathcal{N}(A)$, it follows that for $A_{12} : \mathcal{N}(A) \to \mathcal{R}(A)$ we have $A_{12} = O$, and for $A_{22} : \mathcal{N}(A) \to M$ we have $A_{22} = O$. Also, for $A_{21} : N \to M$, since M is a complement of $\mathcal{R}(A)$, and $Ax \in \mathcal{R}(A)$ for every $x \in N$, then $Ax = 0$ for every $x \in N$, hence $A_{21} = O$. So, we get

$$A = \begin{bmatrix} A_{11} & O \\ O & O \end{bmatrix} : \begin{bmatrix} N \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ M \end{bmatrix}. \tag{4}$$

With respect to the same decomposition,

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ M \end{bmatrix} \to \begin{bmatrix} N \\ \mathcal{N}(A) \end{bmatrix}.$$

Now, since $ABA = A$, from

$$\begin{bmatrix} A_{11} & O \\ O & O \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} A_{11} & O \\ O & O \end{bmatrix} = \begin{bmatrix} A_{11}B_{11}A_{11} & O \\ O & O \end{bmatrix}$$

we have $A_{11}B_{11}A_{11} = A_{11}$, and recalling A_{11} is invertible, we see that $B_{11} = A_{11}^{-1}$.

Since $(BA)^2 = BABA = BA$ and $\mathcal{N}(A) = \mathcal{N}(ABA) \supseteq \mathcal{N}(BA) \supseteq \mathcal{N}(A)$, it follows BA is a projection onto N, thus

$$BA = \begin{bmatrix} I & O \\ O & O \end{bmatrix} : \begin{bmatrix} N \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} N \\ \mathcal{N}(A) \end{bmatrix}.$$

But

$$BA = \begin{bmatrix} A_{11}^{-1} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} A_{11} & O \\ O & O \end{bmatrix} = \begin{bmatrix} A_{11}^{-1}A_{11} & O \\ B_{21}A_{11} & O \end{bmatrix},$$

so $B_{21}A_{11} = O$, and since A_{11} is invertible, it follows $B_{21} = O$.

In a similar way, we saw above that AB is a projection onto $\mathcal{R}(A)$, thus

$$AB = \begin{bmatrix} I & O \\ O & O \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ M \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ M \end{bmatrix}.$$

But

$$AB = \begin{bmatrix} A_{11} & O \\ O & O \end{bmatrix} \begin{bmatrix} A_{11}^{-1} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} A_{11}A_{11}^{-1} & A_{11}B_{12} \\ O & O \end{bmatrix},$$

so $A_{11}B_{12} = O$, and since A_{11} is invertible, it follows $B_{12} = O$. Therefore, we arrive to the following matrix form for B:

$$B = \begin{bmatrix} A_{11}^{-1} & O \\ O & B_{22} \end{bmatrix}$$

where $B_{22} : M \to \mathcal{N}(A)$ is arbitrary.

Thus, we have proved the following theorem.

Theorem : Let $A \in \mathcal{B}(X)$ and suppose that $\mathcal{R}(A)$ and $\mathcal{N}(A)$ are closed and complemented with complements M and N respectively. Then A is inner invertible and for any inner inverse $B \in \mathcal{B}(X)$ we have the following matrix forms:

$$A = \begin{bmatrix} A_1 & O \\ O & O \end{bmatrix} : \begin{bmatrix} N \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ M \end{bmatrix},$$

where A_1 is invertible, and

$$A = \begin{bmatrix} A_1^{-1} & O \\ O & B_2 \end{bmatrix} \begin{bmatrix} \mathcal{R}(A) \\ M \end{bmatrix} \to \begin{bmatrix} N \\ \mathcal{N}(A) \end{bmatrix},$$

with B_2 arbitrary.

Observation

Notice that the theorem above shows that we don't have uniqueness for the inner inverse. Indeed, given an inner inverse for an operator, we construct another inner inverse, although not necessarily distinct, with an interesting property.

Suppose $A = ABA$. Now let $C := BAB$, then $ACA = ABABA = ABA = A$ and $CAC = BABABAB = BABAB = BAB = C$. Thus, C is an inner inverse for A which also satisfies $C = CAC$. We will give this C a name:

<u>Definition :</u> Let $A \in \mathcal{B}(X)$, if there exists $C \in \mathcal{B}(X)$, $C \neq O$, such that $C = CAC$, then C is called an outer inverse for A, and we say that A is outer invertible.

We can say that if A is inner invertible then A is outer invertible.

Construction of an outer inverse for every nonzero operator.

We constructed an inner inverse for A provided its range and null space were closed and complemented. Now we show that we can construct an outer inverse for every nonzero operator.

Theorem : Let $A \in B(X)$ be a nonzero operator, then there exists $C \in B(X), C \neq 0$, such that $C = CAC$.

Proof. Since $A \neq 0$, there exists $x_0 \in X$ such that $Ax_0 \neq 0$, Let $y_0 = Ax_0$, Since span$\{x_0\}$ and span $\{y_0\}$ are finite dimensional, they are complemented. Thus, there exist subspaces M, N such that

$$X = \text{span}\{x_0\} \oplus N = \text{span}\{y_0\} \oplus M.$$

We have the following matrix form for A with respect to these decompositions:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} : \begin{bmatrix} \text{span}\{x_0\} \\ N \end{bmatrix} \to \begin{bmatrix} \text{span}\{y_0\} \\ M \end{bmatrix}.$$

It is clear that $A_{11} : \text{span}\{x_0\} \to \text{span}\{y_0\}$ is invertible. Now, taking

$$A = \begin{bmatrix} A_{11}^{-1} & O \\ O & O \end{bmatrix} : \begin{bmatrix} \text{apan}\{y_0\} \\ M \end{bmatrix} \to \begin{bmatrix} \text{span}\{x_0\} \\ N \end{bmatrix}.$$

we get $CAC = C$.

We have seen that inner invertibility implies outer invertibility. The theorem above says that outer invertibility is more general than inner invertibility.

We assume that for $A \in \mathcal{B}(X)$ there exists $C \in \mathcal{B}(X)$ such that $C = CAC$ holds and $C \neq O$. We are interested in matrix forms for A and C.

As for inner inverses, we have

$$(CA)^2 = CACA = CA,$$

$$(AC)^2 = ACAC = AC.$$

Also, from $\mathcal{R}(C) = \mathcal{R}(CAC) \subseteq \mathcal{R}(CA) \subseteq \mathcal{R}(C)$ we have

$$\mathcal{R}(C) = \mathcal{R}(CA);$$

and from $\mathcal{N}(C) = \mathcal{N}(CAC) \supseteq \mathcal{N}(AC) \supseteq \mathcal{N}(C)$ we have

$$\mathcal{N}(C) = \mathcal{N}(AC).$$

Thus, $\mathcal{R}(C)$ and $\mathcal{N}(C)$ are closed and complemented.

Let $M := \mathcal{R}(C), M_1 := \mathcal{N}(CA)$, and $N := \mathcal{N}(C)$, then $\mathcal{R}(AC) = A(\mathcal{R}(C)) = A(M)$ and

$$X = M \oplus M_1 = A(M) \oplus N.$$

Let us consider the following matrix form with respect to these decompositions:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} : \begin{bmatrix} M \\ M_1 \end{bmatrix} \to \begin{bmatrix} A(M) \\ N \end{bmatrix}$$

It is clear that A_{11} is onto; to see that it is also $1-1$, let $x \in M$ such that $Ax = 0$, since $M = \mathcal{R}(CA)$, there is some y such that $x = CAy$, then $0 = CAx = CACAy = CAy = x$.

For $A_{12} : M_1 \to A(M)$, if $x \in M_1 = \mathcal{N}(CA)$, then $CAx = 0$, it follows that $Ax \in \mathcal{N}(C)$, and since $\mathcal{N}(C) \cap A(M) = \mathcal{N}(AC) \cap \mathcal{R}(AC) = \{0\}$, we have that $Ax = 0$ and $A_{12} = O$.

Finally, for $A_{21} : M \to N$, if $x \in M = \mathcal{R}(C)$, then there exists y such that $x = Cy$, hence $Ax = ACy \in \mathcal{R}(AC)$, and since $N \cap \mathcal{R}(AC) = \mathcal{N}(AC) \cap \mathcal{R}(AC) = \{0\}$, we have $Ax = 0$ and $A_{21} = O$. Thus,

$$A = \begin{bmatrix} A_{11} & O \\ O & A_{22} \end{bmatrix}$$

with A_{11} invertible and A_{22} arbitrary.

Now consider the following matrix form of C with respect to the same (fixed) decompositions:

$$C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} : \begin{bmatrix} A(M) \\ N \end{bmatrix} \to \begin{bmatrix} M \\ M_1 \end{bmatrix}.$$

From $C = CAC$ we have that A is an inner inverse for C, and from the results for inner inverses we have

$$C = \begin{bmatrix} A_{11}^{-1} & O \\ O & O \end{bmatrix}.$$

The outer inverse is not unique, in general. However, the matrix form of C above shows that the outer inverse is unique when we fix its range and null space.

Thus, we have proved :

<u>Theorem :</u> Let $\mathcal{A} \in \mathcal{B}(X)$ be a nonzero operator and M, N subspaces of X. If $C \in \mathcal{B}(X)$ is an outer inverse for A such that $\mathcal{R}(C) = M$ and $\mathcal{N}(C) = N$, then we have the following matrix forms:

$$A = \begin{bmatrix} A_1 & O \\ O & A_2 \end{bmatrix} : \begin{bmatrix} M \\ M_1 \end{bmatrix} \to \begin{bmatrix} A(M) \\ N \end{bmatrix},$$

with A_1 invertible and A_2 arbitrary, and

$$C = \begin{bmatrix} A_1^{-1} & O \\ O & O \end{bmatrix} : \begin{bmatrix} A(M) \\ N \end{bmatrix} \to \begin{bmatrix} M \\ M_1 \end{bmatrix}.$$

A Class of Outer Inverse

We saw above that an outer inverse is unique if we fix its range and null space. We now fix these subspaces by means of another operator.

<u>Definition :</u> Let $A, T \in \mathcal{B}(X)$ be nonzero operators. If there exists an outer inverse C for A such that $\mathcal{R}(C) = \mathcal{R}(T)$ and $\mathcal{N}(C) = \mathcal{N}(T)$, then we say that A is invertible along T, and we write $C = A^{-T}$.

Notice that A is invertible if and only if it is invertible along I, and the inverse is A^{-I}. Since we are fixing the range and null space of an outer inverse, the inverse along an operator is unique if it exists.

We can give a characterization of the set of operators along which an operator A is invertible:

A Class of Outer Inverse

<u>Theorem :</u> Let $A, T \in \mathcal{B}(X)$ be nonzero operators. The following statements are equivalent.

(1) A is invertible along T.

(2) $\mathcal{R}(T)$ is closed and complemented subspace of $X, A(\mathcal{R}(T)) = \mathcal{R}(AT)$ is closed such that $\mathcal{R}(AT) \oplus \mathcal{N}(T) = X$ and the reduction $A|_{\mathcal{R}(T)} : \mathcal{R}(T) \to \mathcal{R}(AT)$ is invertible.

We are interested in refining the matrix forms used in the above theorem. If A is invertible along T with $C = A^{-T}$, then A is outer invertible and A has the following matrix form:

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(T) \\ \mathcal{N}(CA) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(AC) \\ \mathcal{N}(T) \end{bmatrix},$$

with A_1 invertible.

Notice that, since $\mathcal{R}(T)$ and $\mathcal{N}(T)$ are closed and complemented (because C is inner invertible), T is inner invertible, and

$$T = \begin{bmatrix} T_1 & O \\ O & O \end{bmatrix} : \begin{bmatrix} \mathcal{R}(AC) \\ \mathcal{N}(T) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(T) \\ \mathcal{N}(CA) \end{bmatrix},$$

with T_1 invertible.

Now, we would like to have the matrix forms in terms of A an T only. From the matrix forms

$$TA = \begin{bmatrix} T_1A_1 & O \\ O & O \end{bmatrix} : \begin{bmatrix} \mathcal{R}(T) \\ \mathcal{N}(CA) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(T) \\ \mathcal{N}(CA) \end{bmatrix},$$

$$AT = \begin{bmatrix} T_1A_1 & O \\ O & O \end{bmatrix} : \begin{bmatrix} \mathcal{R}(AC) \\ \mathcal{N}(T) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(AC) \\ \mathcal{N}(T) \end{bmatrix},$$

since T_1 and A_1 are invertible, it follows that $\mathcal{N}(TA) = \mathcal{N}(CA)$ and $\mathcal{R}(AT) = \mathcal{R}(AC)$. Thus, we have arrived to the following:

Theorem : Let $A, T \in \mathcal{B}(X)$. If A is invertible along T, then we have the following matrix forms for A, T and A^{-T} with respect to the decomposition $X = \mathcal{R}(T) \oplus \mathcal{N}(TA) = \mathcal{R}(AT) \oplus \mathcal{N}(T)$:

$$A = \begin{bmatrix} A_1 & O \\ O & A_2 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(T) \\ \mathcal{N}(TA) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(AT) \\ \mathcal{N}(T) \end{bmatrix}, \quad (A_1 \text{invertible})$$

$$T = \begin{bmatrix} T_1 & O \\ O & O \end{bmatrix} : \begin{bmatrix} \mathcal{R}(AT) \\ \mathcal{N}(T) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(T) \\ \mathcal{N}(TA) \end{bmatrix}, \quad (T_1 \text{invertible})$$

and

$$A^{-T} = \begin{bmatrix} A_1^{-1} & O \\ O & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(AT) \\ \mathcal{N}(T) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(T) \\ \mathcal{N}(TA) \end{bmatrix},$$

Conclusion and Final Remarks

In a Hilbert space, every closed subspace is complemented (by its orthogonal complement), so every closed range operator on a Hilbert space is inner invertible.

If we require the operator $A \in \mathcal{B}(X)$ to be inner and outer invertible, we still cannot guarantee uniqueness. However, if there exists $B \in \mathcal{B}(X)$ such that $X = ABA$ and $AB = BA$, then taking $C = BAB$ we have $A = ACA$, $C = CAC$ and $CA = AC$, and this C is unique. This C is called the "group inverse".

Since inner invertibility implies outer invertibility, it is natural to weaken inner invertiblility while requiring outer invertibility. If A is outer invertible with outer inverse B such that $BA = AB$ and there exists n such that $A = A^n BA$, then A is said to be "Drazin invertible", and the least n such that $A = A^n BA$ holds is called the Drazin index of A.

References

- D.S. Djordjevic and V. Rakocevic, Lectures on generalized inverses, University of Nis, Faculty of Sciences and Mathematics, Nis, 2008.
- Kantún-Motiel, G, Matrix representations of inner and outer inverses. RIMS Kôkyûroku (1893), 13-19, 2014.